# Floating point

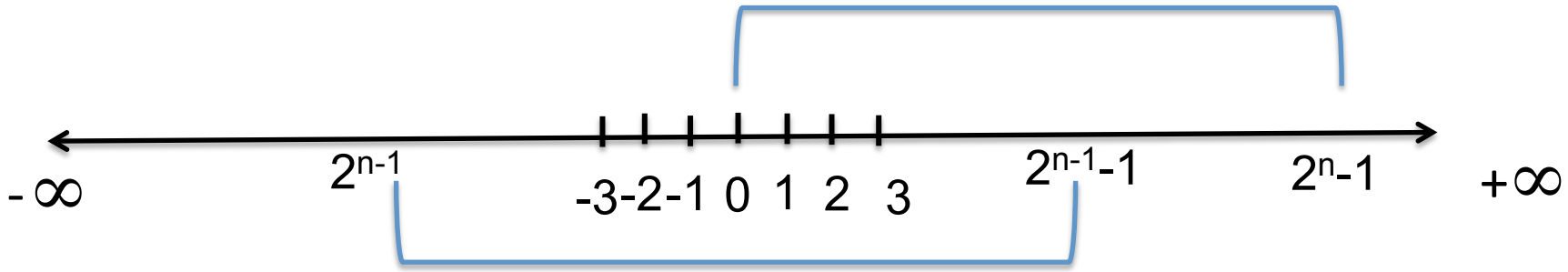
Jinyang Li

based on Tiger Wang's slides

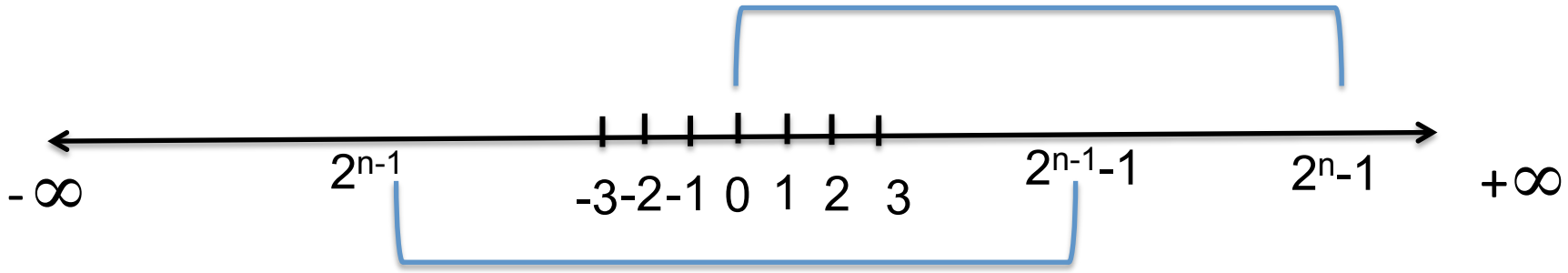
# Representing Real Numbers using bits

# Representing Real Numbers using bits

## What we have studied



# Representing Real Numbers using bits




Today: How to represent fractional numbers?

# Decimal Representation

| Real Numbers | Decimal Representation (Expansion) |
|---|---|
| 11 / 2 | $(5.5)_{10}$ |
| 1 / 3 | $(0.3333333...)_{10}$ |
| $\sqrt{2}$ | $(1.4128\ldots)_{10}$ |

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$(5.5)_{10} = 5 * 10^{0} + 5 * 10^{-1}$

$(0.333333...)_{10} = 3 * 10^{-1} + 3 * 10^{-2} + 3 * 10^{-3} + ...$

$(1.4128...)_{10} = 1 * 10^{0} + 4 * 10^{-1} + 1 * 10^{-2} + 2 * 10^{-3} + ...$

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$$r_{10} = (d_m d_{m-1} \ldots d_1 d_0 \cdot d_{-1} d_{-2} \ldots d_{-n})_{10}$$

$$= \sum_{i=-n}^{m} 10^i \times d_i$$

# Binary Representation

$(5.5)_{10} = 4 + 1 + 1 / 2$

$= 1 * 2^{2} + 1 * 2^{0} + 1 * 2^{-1}$

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$= 1 * 2^2 + 0 * 2^1 + 1 * 2^0 + 1 * 2^{-1}$

$= (101.1)_2$

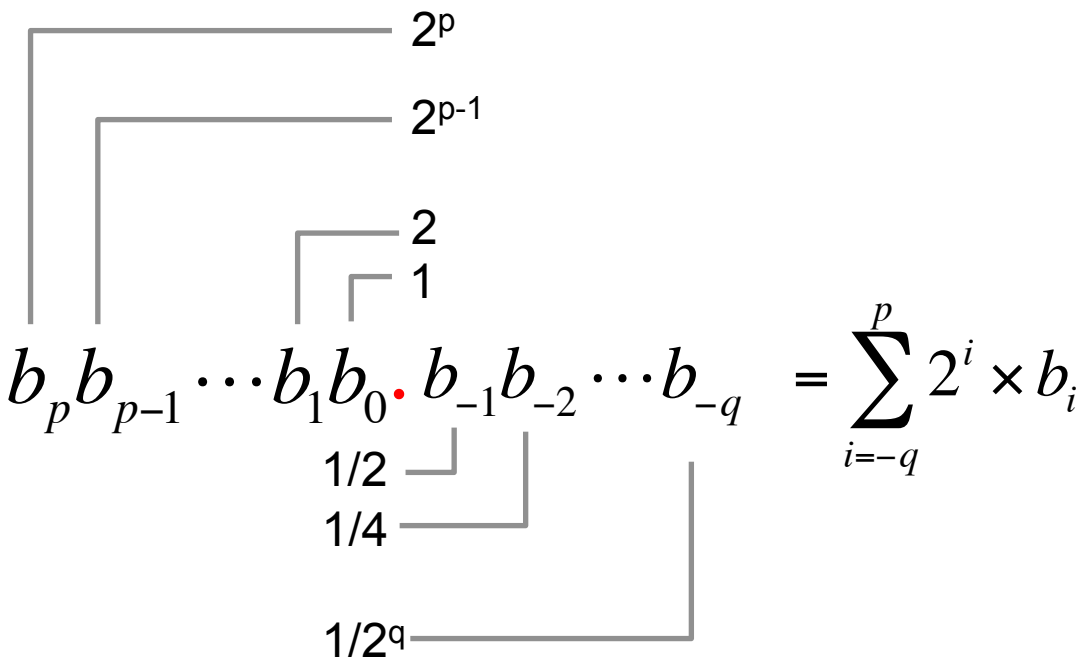
$(0.333333...)_{10} = 1 / 4 + 1 / 16 + 1 / 64 + ...$

$= (0.01010101...)_2$

# Binary Representation

$$r_{10} = (d_m d_{m-1} d_1 d_0 \bullet d_{-1} d_{-2} \ldots d_{-n})_{10}$$

$$= (b_p b_{p-1} b_1 b_0 \bullet b_{-1} b_{-2} \ldots b_{-q})_2$$



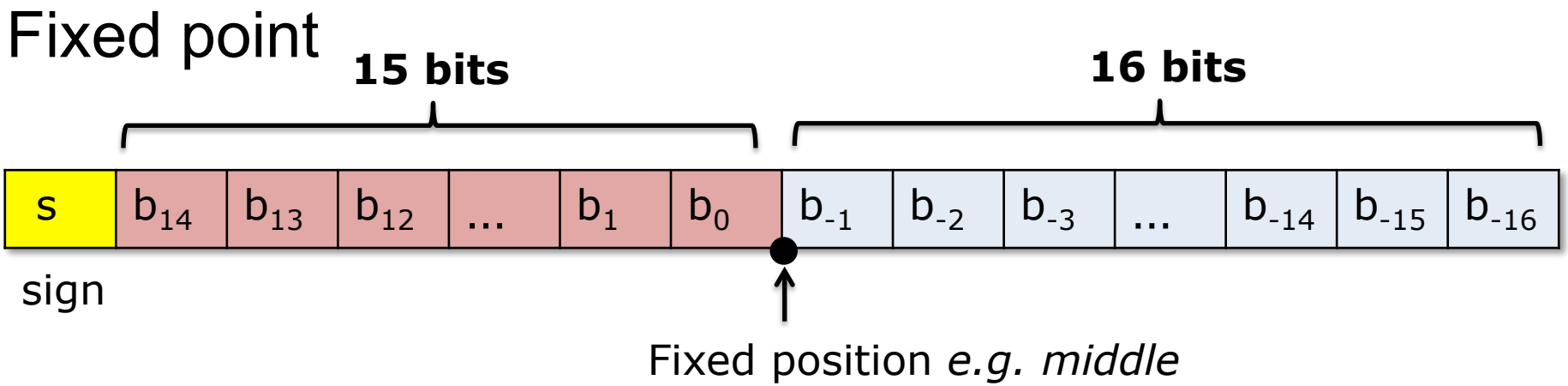
# Exercise

| Binary Expansion | Formula | Decimal |
| --- | --- | --- |
| $10.011_2$ | | |
| | $2^{-3} + 2^{-4} + 2^{-6}$ | |
| | $2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}$ | |

# Exercise

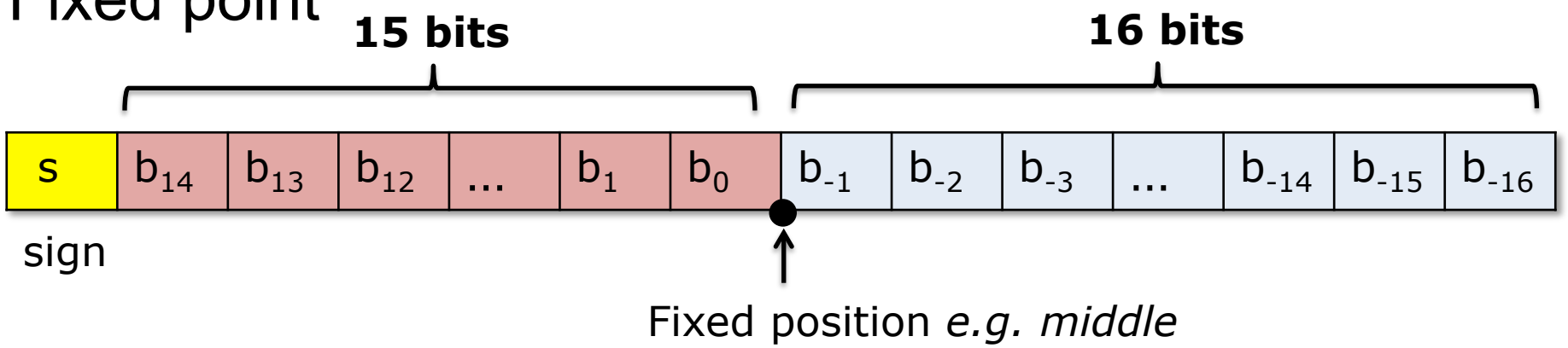
| Binary Expansion | Formula | Decimal |
|---|---|---|
| $10.011_2$ | $2^1 + 2^{-2} + 2^{-3}$ | $2.375_{10}$ |
| $0.001101_2$ | $2^{-3} + 2^{-4} + 2^{-6}$ | $0.203125_{10}$ |
| $0.1111_2$ | $2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}$ | $0.9375_{10}$ |

# Intuitive Idea

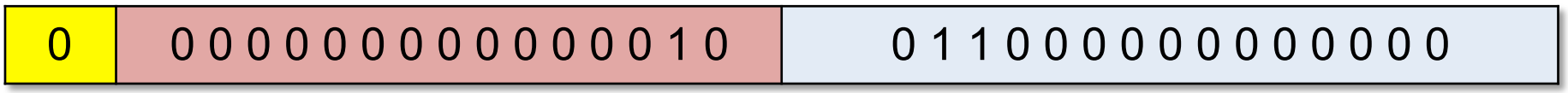


# Intuitive Idea

Fixed point




$( 10.011 )_2$



# Problems of Fixed Point

Limited range and precision: e.g., 32 bits

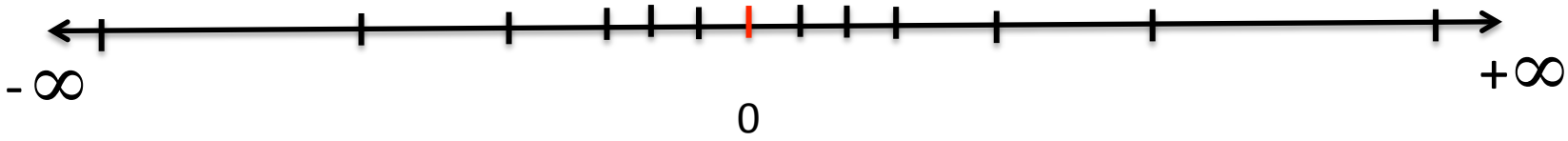
- Largest number: $2^{15}\ (011\ldots111)_2$
- Highest precision: $2^{-16}$

→Rarely used (No built-in hardware support)

# The idea

- Limitation of fixed point notation:
  - Represents evenly spaced fractional numbers
    - ➔ hard tradeoff between high precision and high magnitude

- How about un-even spacing between numbers?



# Floating Point: decimal

Based on exponential notation (aka normalized scientific notation)

$r_{10} = \pm M * 10^{E}$, where $1 <= M < 10$

M: significant (mantissa), E: exponent

# Floating Point: decimal

Example:

$365.25 = 3.6525 * 10^{2}$

$0.0123 = 1.23 * 10^{-2}$

Decimal point **floats** to the position immediately after the first nonzero digit.

# Floating Point: binary

Binary exponential representation

$r_{10} = \pm M * 2^E$, where $1 <= M < 2$

$M = ( 1.b_1b_2b_3\ldots b_n )_2$

M: significant, E: exponent

$(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

# Floating Point

Binary exponential representation

$r_{10} = \pm M * 2^E$, where $1 <= M < 2$

$M = ( 1.b_1 b_2 b_3 \ldots b_n )_2$

} Normalized representation of r

M: significant, E: exponent

$(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

Normalization: give a number r, obtain its normalized representation

# Exercises

The normalized representation of $(10.25)_{10}$ is ?

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The normalized representation of $(10.25)_{10}$ is ?

$$(10.25)_{10} = (1010.01)_2 = (1.01001)_2 * 2^3$$

# Floating Point

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$(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

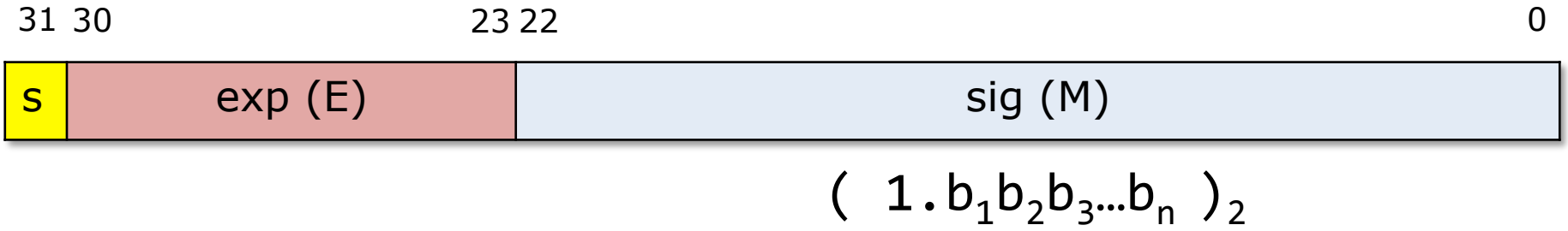
**How to represent a normalized number?**

# Normalized representation

$r_{10} = \pm M * 2^E$, where $1 <= M < 2$

$M = ( 1.b_1b_2b_3\ldots b_n )_2$

M: significant, E: exponent

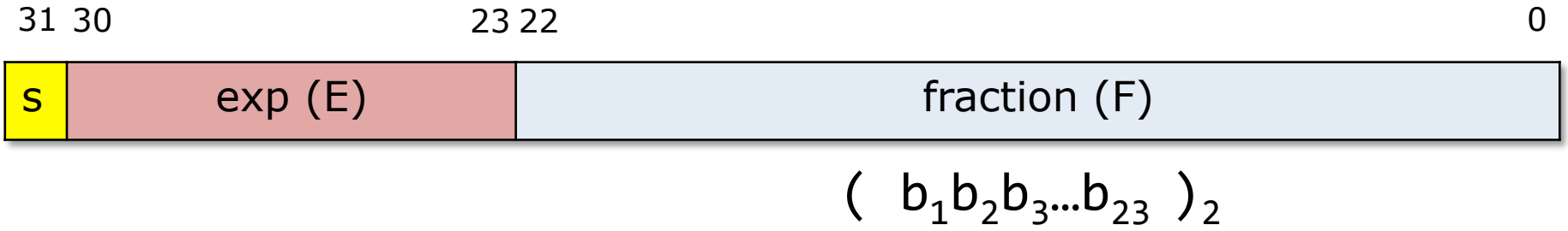


# Normalized representation in computer

$r_{10} = \pm M * 2^{E}$, where $1 <= M < 2$

$M = ( 1.b_1b_2b_3\ldots b_{23} )_2$

M: significant, E: exponent



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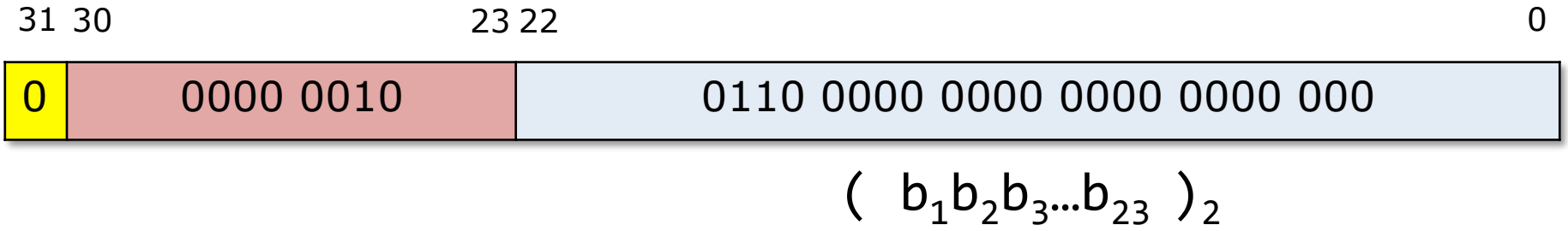


$(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

# Exercise

Given the normalized representation of $(71)_{10}$ and $(10.25)_{10}$

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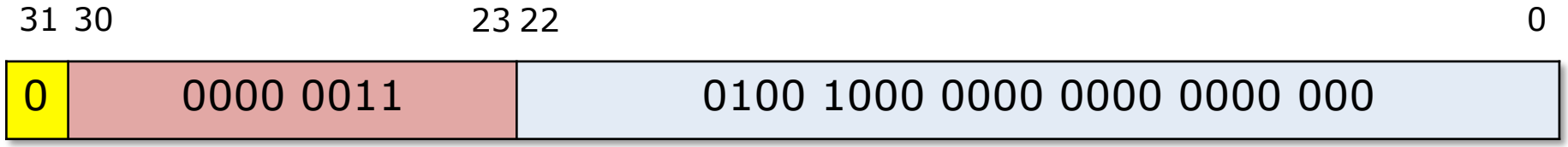
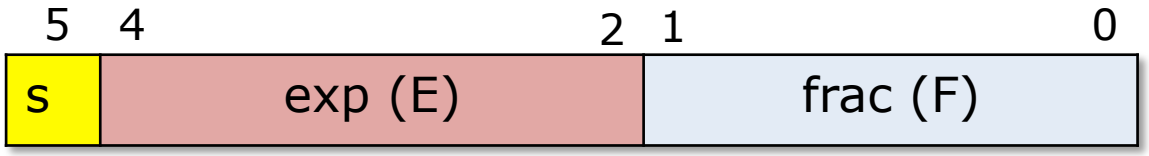
Given the normalized representation of $(71)_{10}$ and $(10.25)_{10}$

$(10.25)_{10} = (1010.01)_2 = (1.01001)_2 * 2^3$




$(71)_{10} = (1000111)_2 = (1.000111)_2 * 2^6$

# Toy Number System

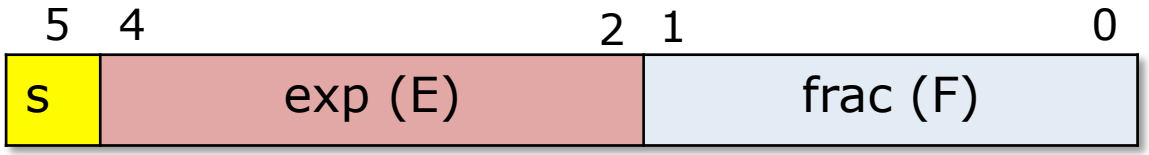



6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits

**Largest positive number ?**

# Toy Number System




6-bit floating point representation

- exponent: 3 bits
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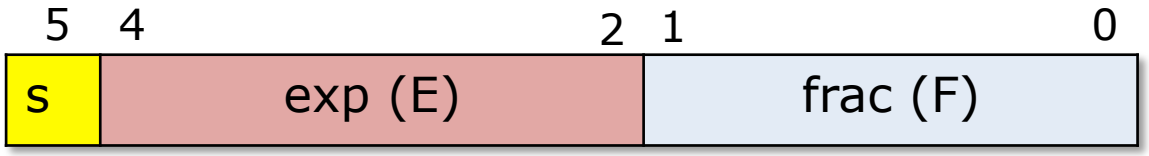
**Largest positive number ?**


$( 1.11 )_2 * 2^7 = 224$

# Toy Number System




6-bit floating point representation

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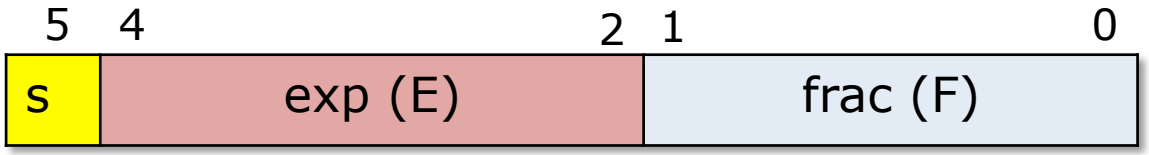
**Largest positive number: 224**

**Smallest positive number ?**

# Toy Number System




6-bit floating point representation

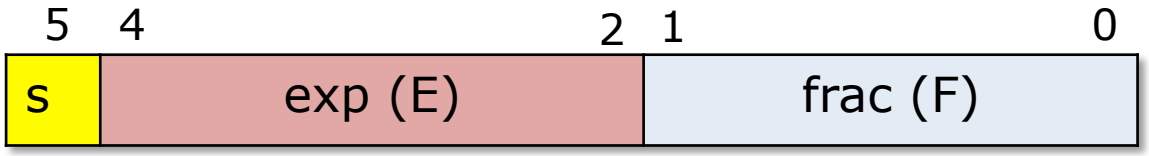
- exponent: 3 bits
- fraction: 2 bits

**Largest positive number: 224**

**Smallest positive number: 1**


$( 1.00 )_2 * 2^0 = 1$

# Toy Number System




6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits

**Positive number: 1 to 224**

**Negative number: -224 to -1**

No more bit patterns left to represent numbers
(-1, 1)

# Questions

How to represent

1. number close or equal to 0?
2. larger numbers, even $\infty$ ?
3. the result of dividing by 0 ?

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How to represent

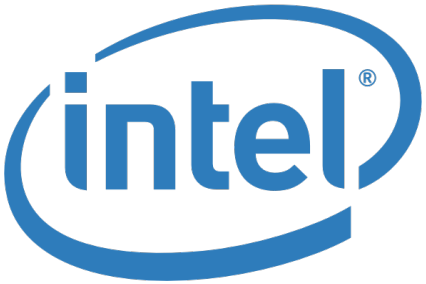
1. number close or equal to 0?
2. larger numbers, even $\infty$ ?
3. the result of dividing by 0 ?

**Lots of different implementations around 1950s!**

# IEEE Floating Point Standard

IEEE p754
A standard for binary
floating point representation

Prof. William Kahan
University of California at Berkeley
Turing Award (1989)







# The Only Book Focuses On IEEE Floating Point Standard

## Numerical Computing with IEEE Floating Point Arithmetic

**Including One Theorem, One Rule of Thumb, and One Hundred and One Exercises**

**Michael L. Overton**
Courant Institute of Mathematical Sciences
New York University
New York, New York

hardware. This degree of altruism was so astonishing that MATLAB's creator Cleve Moler used to advise foreign visitors not to miss the country's two most awesome spectacles: the Grand Canyon, and meetings of IEEE p754."

https://cs.nyu.edu/overton/NumericalComputing/protected/NumericalComputingSIAM.pdf
With you nyu netid/password. You can also search the pdf with google.

# Goals of IEEE Standard

Consistent representation of floating point numbers by all machines adopting the standard

Correctly rounded floating point operations, using several rounding modes, since exact answers often cannot be represented exactly on the computer

Consistent treatment of exceptional situations such as division by zero

# Restrictions for Normalized Representation

$r_{10} = \pm M * 2^E$, where $1 <= M < 2$

$M = ( 1.b_0b_1b_2b_3\ldots b_n )_2$

M: significant, E: exponent


$( b_0b_1b_2b_3\ldots b_n )_2$

E can not be $(1111\ 1111)_2$ or $(0000\ 0000)_0$

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$E_{max} = ?$

$E_{min} = ?$

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$( b_0b_1b_2b_3\ldots b_n )_2$

E can not be $(1111\ 1111)_2$ or $(0000\ 0000)_0$

$E_{max} = 254, (1111\ 1110)_2$

$E_{min} = 1, (0000\ 0001)_2$

**represent values between -1 and 1 in normalized representation**

# Exponential Bias

$r_{10} = \pm M * 2^E$, where $1 <= M < 2$

$M = ( 1.b_0b_1b_2b_3\ldots b_n )_2$

M: significant, E: exponent


$( b_0b_1b_2b_3\ldots b_n )_2$

Bias: 127

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$( b_0b_1b_2b_3\ldots b_n )_2$

Bias: 127

$E_{max} =$ ?

$E_{min} =$ ?

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M: significant, E: exponent


$( b_0b_1b_2b_3\ldots b_n )_2$

Bias: 127

$E_{max} = 254 - 127 = 127$

$E_{min} = 1 - 127 = -126$

# Exponential Bias

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$M = ( 1.b_0b_1b_2b_3\ldots b_n )_2$

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Bias: 127

$E_{max} = 254 - 127 = 127$

$E_{min} = 1 - 127 = -126$

Smallest positive number: ?

Negative number with smallest absolute value: ?

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$r_{10} = \pm M * 2^E$, where $1 <= M < 2$

$M = ( 1.b_0b_1b_2b_3\ldots b_n )_2$

M: significant, E: exponent


$( b_0b_1b_2b_3\ldots b_n )_2$

Bias: 127

$E_{max} = 254 - 127 = 127$

$E_{min} = 1 - 127 = -126$

Smallest positive number: $2^{-126}$

Negative number with smallest absolute value: $-2^{-126}$

# Questions from Munachiso

Q1. Why does it need **bias**?

Q2. Why is the **bias** 127?

# Questions from Munachiso

Q1. Why does it need **bias**?

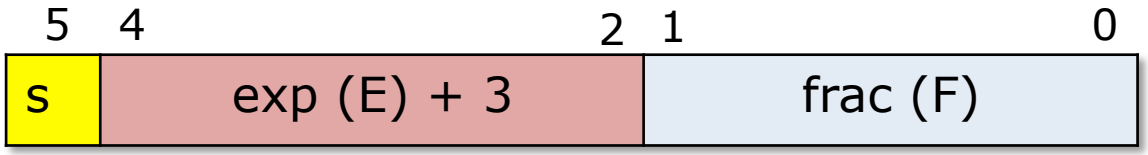
A1. Use unsigned number to represent negative numbers (-1 ~ -126)

# Questions from Munachiso

Q2. Why is 127?

A2. Balance positive numbers (magnitude) and negative numbers (precision)

# Toy Number System

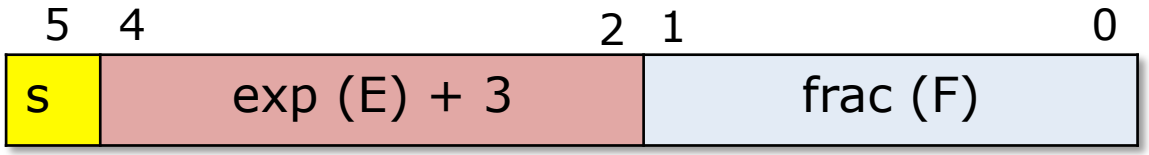



6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits
- **bias: 3**

**Smallest positive number ?**

# Toy Number System



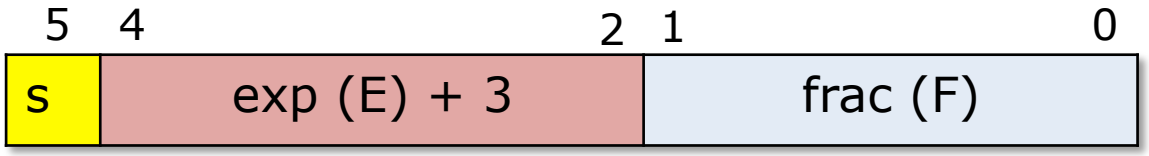

6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits
- **bias: 3**

**Smallest positive number: 0.25**


$( 1.00 )_2 * 2^{-2} = 0.25$

# Toy Number System

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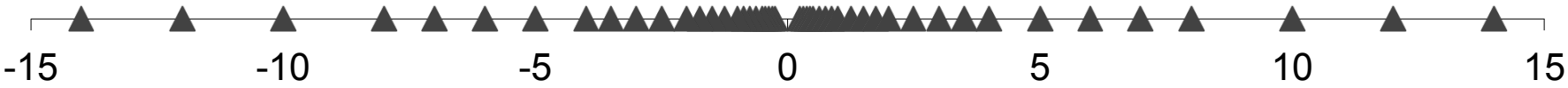

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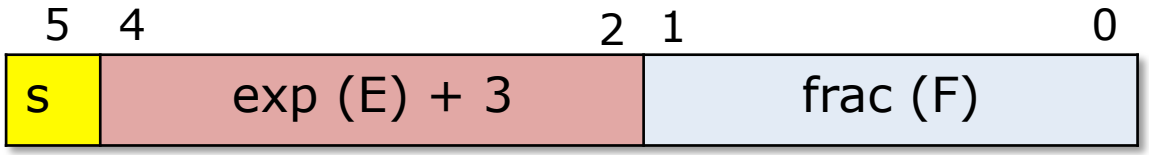


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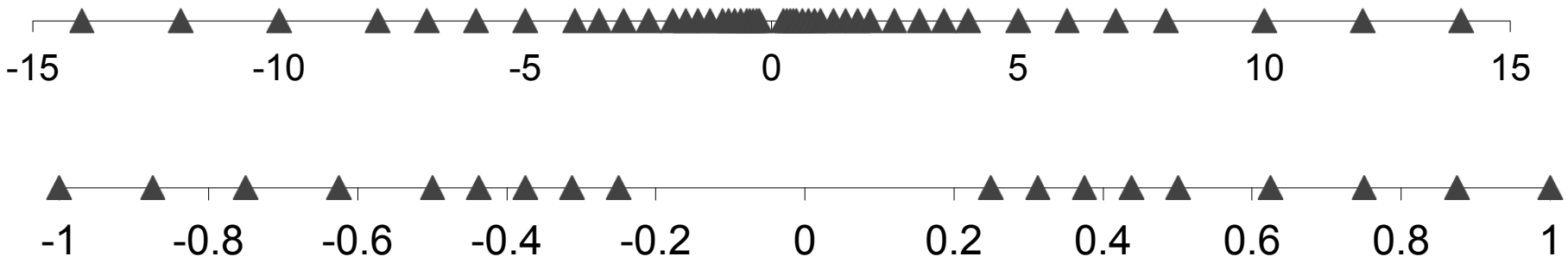

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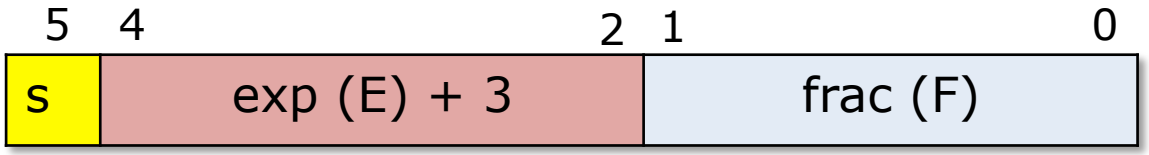


6-bit floating point representation

- exponent: 3 bits
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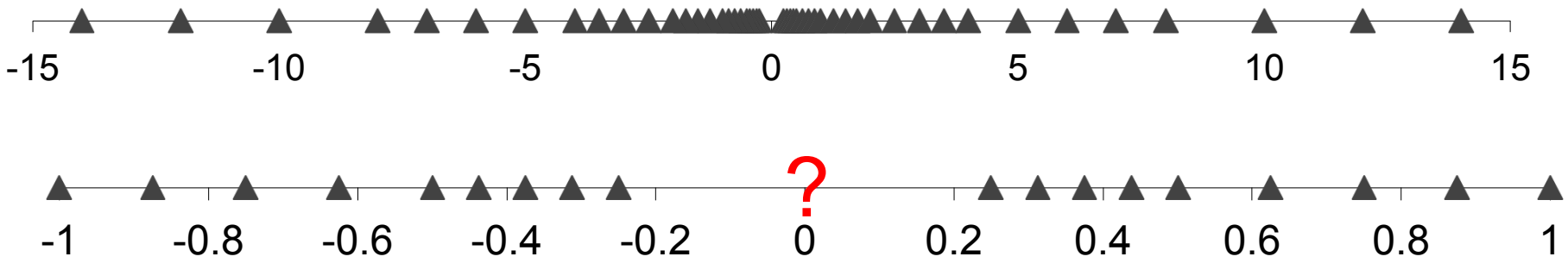
- **bias: 3**



**represent values which are close and equal to 0**

# Denormalization

$r_{10} = \pm M * 2^{E}$, M: significant, E: exponent

**Normalized Encoding:**


$1 <= M < 2, M = ( 1.F )_2$

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$r_{10} = \pm M * 2^E$, M: significant, E: exponent

**Normalized Encoding:**


$1 <= M < 2, M = ( 1.F )_2$

**Denormalized Encoding:**


$E = 1 – Bias = -126$

$0 <= M < 1, M = ( 0.F )_2$

# Zeros

+0.0


0
0000 0000
0000 0000 0000 0000 0000 000

-0.0

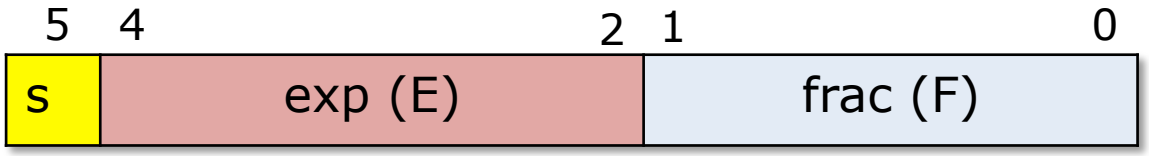

1
0000 0000
0000 0000 0000 0000 0000 000

# Examples

$(0.1)_2 * 2^{-126}$

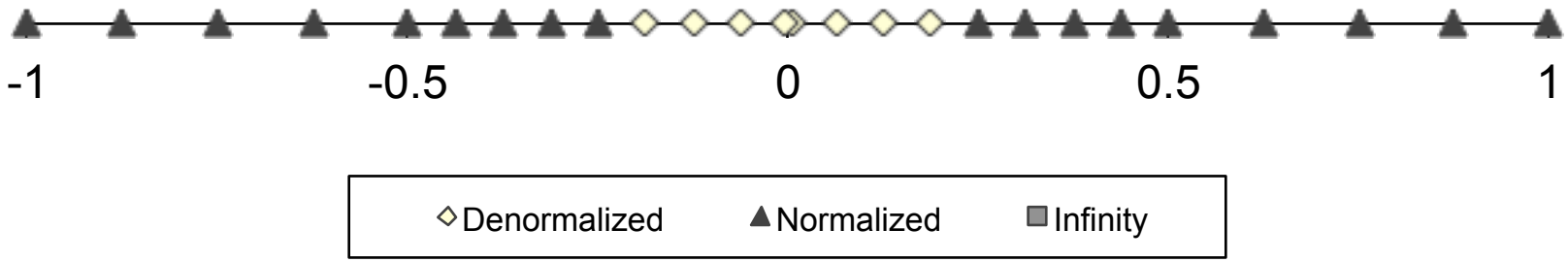

$-(0.010101)_2 * 2^{-126}$

# Toy Number System




6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits
- bias: 3
- **Denormalized encoding**



# Special Values

**Special Value's Encoding:**


| values | sign | frac |
|---|---|---|
| +∞ | 0 | all zeros |
| - ∞ | 1 | all zeros |
| NaN | any | non-zero |

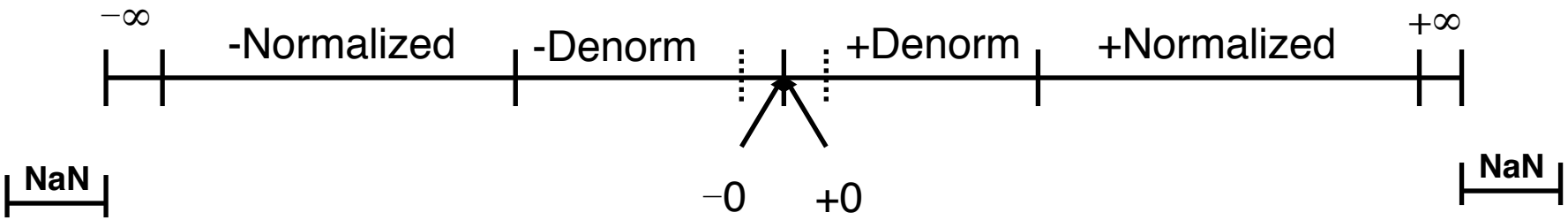
# Exercises

| representation | E | M | V |
|---|---|---|---|
| 0100 1001 0101 0000<br>0000 0000 0000 0000 | | | |
| | | | $2.5 * 2^{-127}$ |
| | | | $-1.25 * 2^{-111}$ |
| 1111 1111 1111 1111<br>0000 0000 0000 0000 | | | |
| 1111 1111 1000 0000<br>0000 0000 0000 0000 | | | |
| | | | $1.5 * 2^{-127}$ |

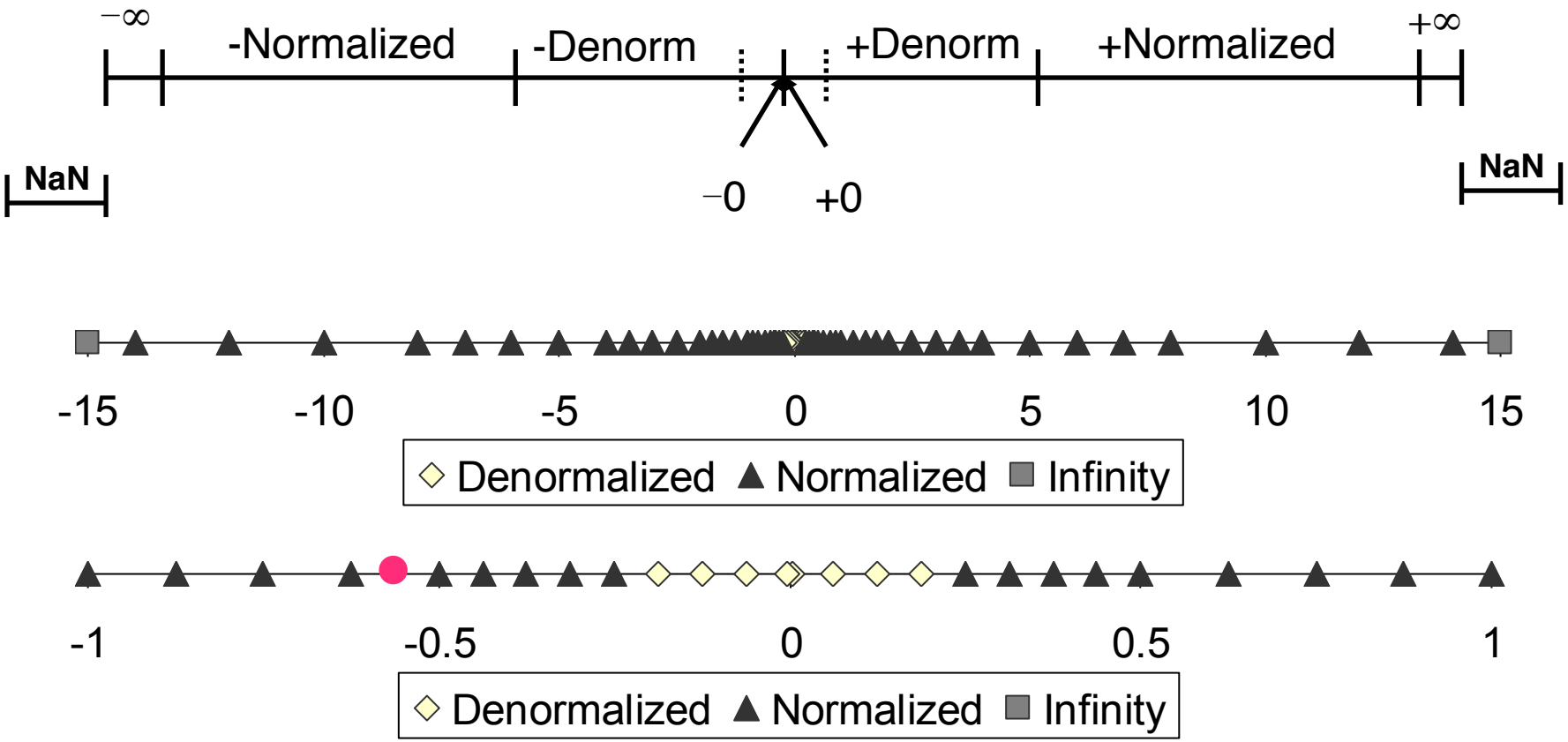
# Exercises

| representation | E | M | V |
|---|---|---|---|
| 0100 1001 0101 0000 0000 0000 0000 0000 | 146 – 127 = 19 | $(1.101)_2$ = 1.625 | $1.625 * 2^{19}$ |
| 0000 0000 1010 0000 0000 0000 0000 0000 | 1 – 127 = -126 | $(1.01)_2$ = 1.25 | $2.5 * 2^{-127}$ = $(1.01)_2 * 2^{-126}$ |
| 1000 1000 0010 0000 0000 0000 0000 0000 | 16 – 127 = -111 | $(1.01)_2$ = 1.125 | $-1.25 * 2^{-111}$ |
| 1111 1111 1111 1111 0000 0000 0000 0000 | - | - | Nan |
| 1111 1111 1000 0000 0000 0000 0000 0000 | - | - | $-\infty$ |
| 0000 0000 0110 0000 0000 0000 0000 0000 | -126 | $(0.11)_2$ | $(0.11)_2 * 2^{-126}$ = $1.5 * 2^{-127}$ |

# Distribution of Representable Values



# Distribution of Representable Values




How to represent the point ● in the format ?

# Rounding

Goal

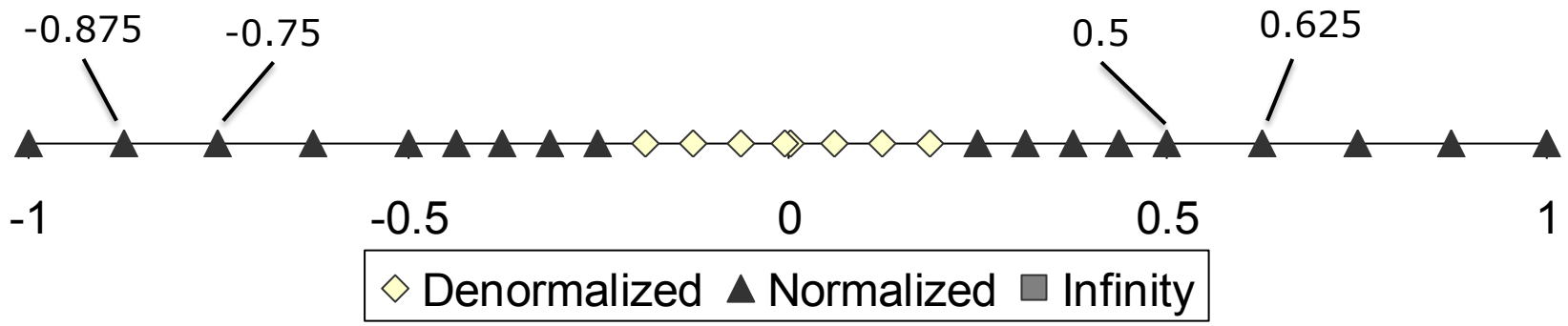
- Given a value *x*, finding the "closest" matching representable value *x'*.

Round modes

- Round-down
- Round-up
- Round-toward-zero
- Round-to-nearest (Round to even in text book)

# Round down

Round(x) = x_ (x_ <= x)

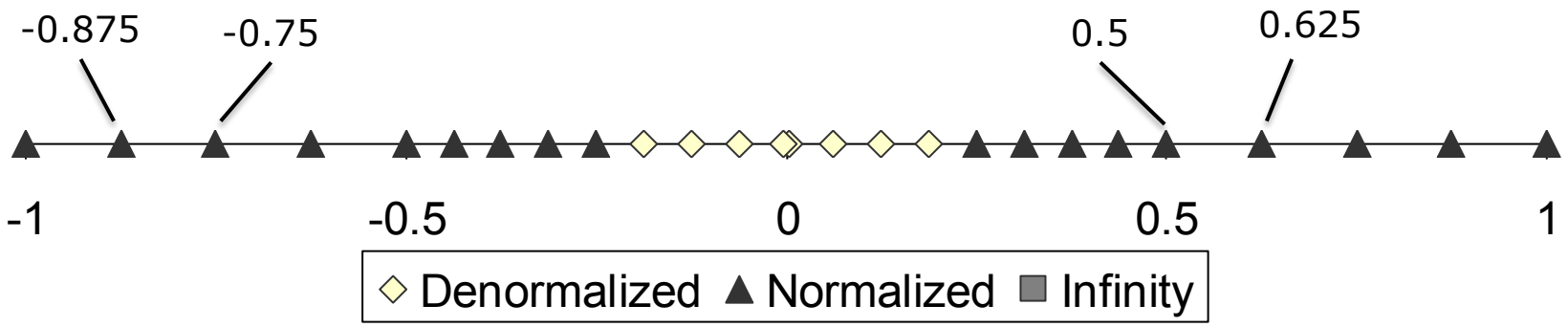



Round(-0.86) = ?

Round(0.55) = ?

# Round down

Round(x) = $x_-$ ($x_-$ <= x)




Round(-0.86) = -0.875

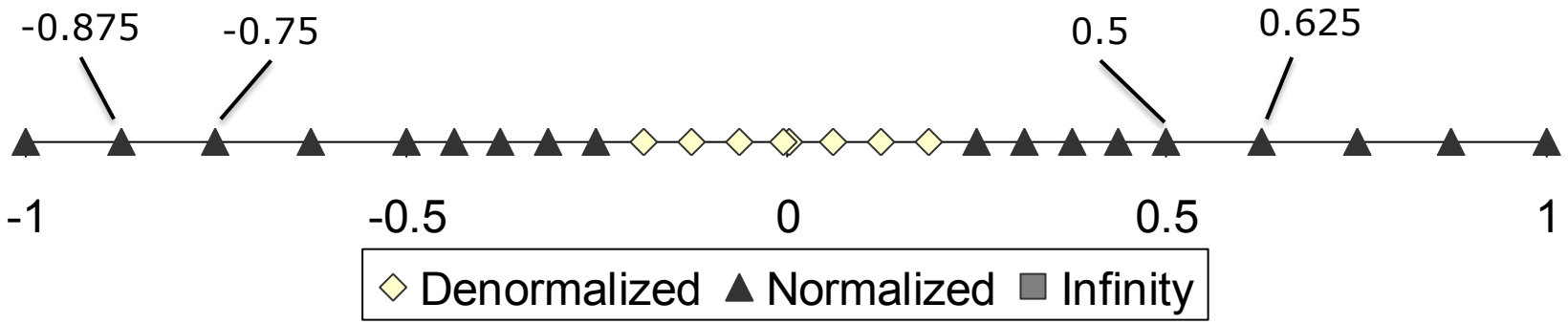
Round(0.55) = 0.5

# Round up

Round(x) = $x_+$  ($x_+$>= x)

# Round up

Round(x) = $x_+$  ($x_+$>= x)

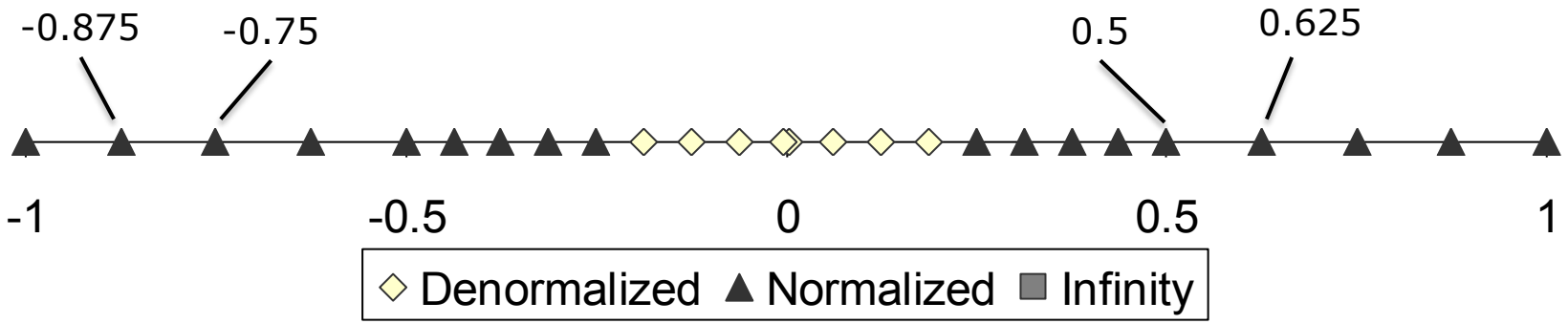



Round(-0.86) = ?

Round(0.55) = ?

# Round up

$Round(x) = x_+ \quad (x_+ >= x)$




Round(-0.86) = -0.75

Round(0.55) = 0.625

# Round towards zero

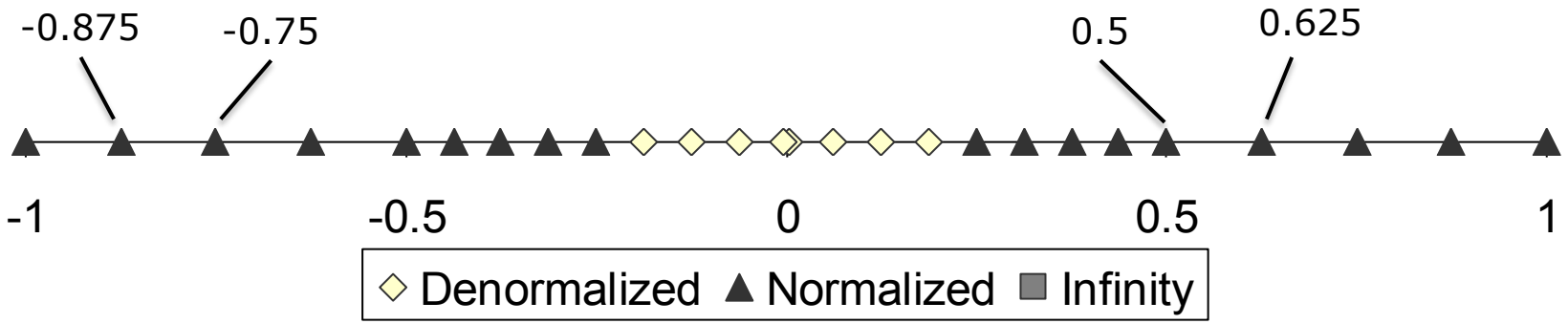
Round(x) = $x_+$ if $x < 0$

Round(x) = $x_-$ if $x > 0$

# Round towards zero

Round(x) = $x_+$ if x < 0

Round(x) = $x_-$ if x > 0



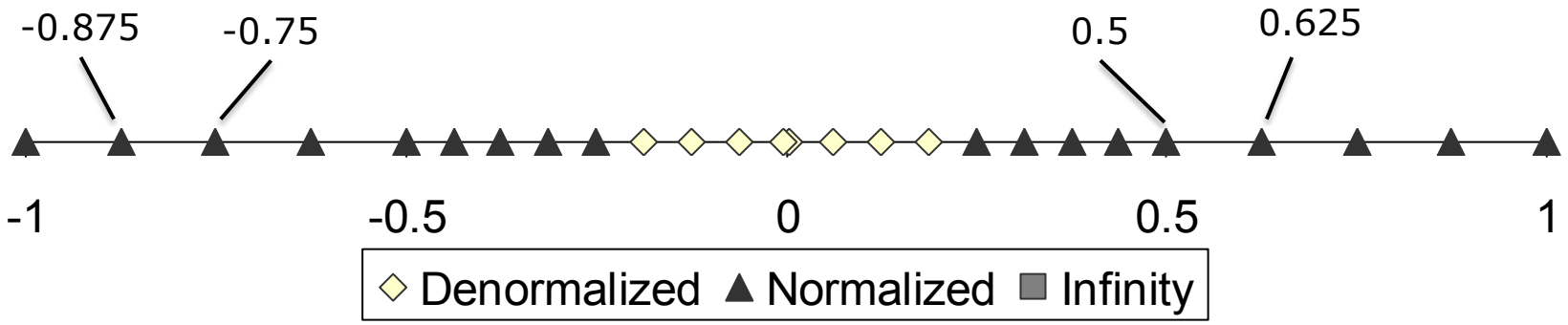

Round(-0.86) = ?

Round(0.55) = ?

# Round towards zero

Round(x) = $x_+$ if x < 0

Round(x) = $x_-$ if x > 0




Round(-0.86) = -0.75

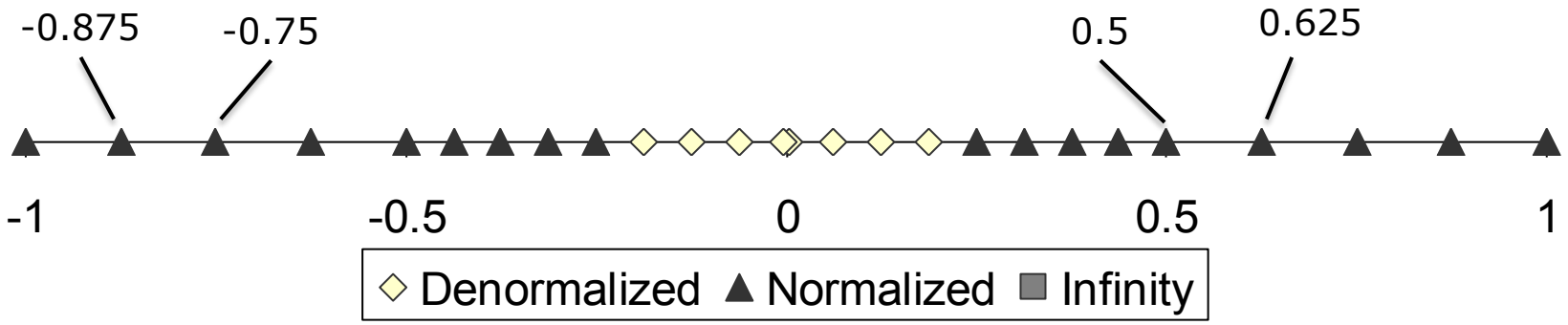
Round(0.55) = 0.5

# Round to nearest

Round(x) either $x_+$ or $x_-$ , whichever is nearer to x.

# Round to nearest

Round(x) either $x_+$ or $x_-$ , whichever is nearer to x.

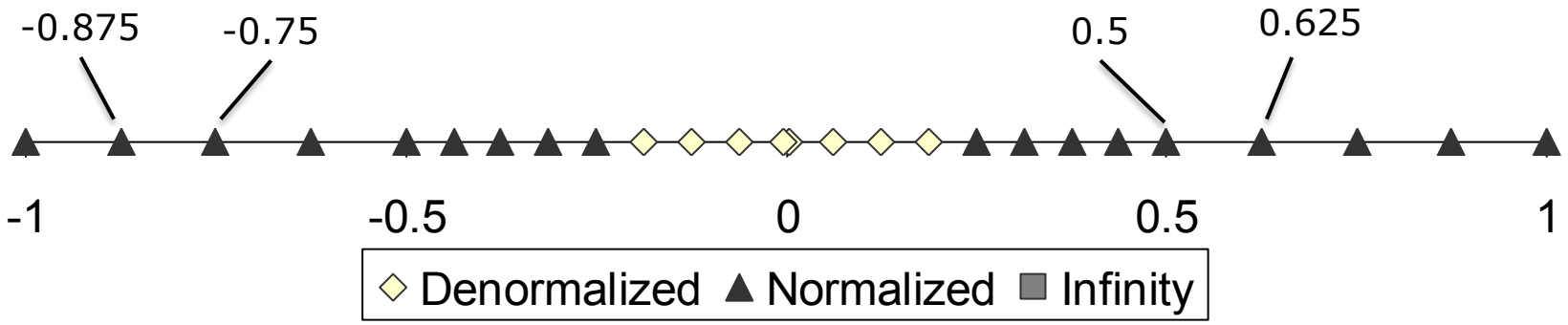



Round(-0.86) = ?

Round(0.55) = ?

# Round to nearest

Round(x) either $x_+$ or $x_-$ , whichever is nearer to x.

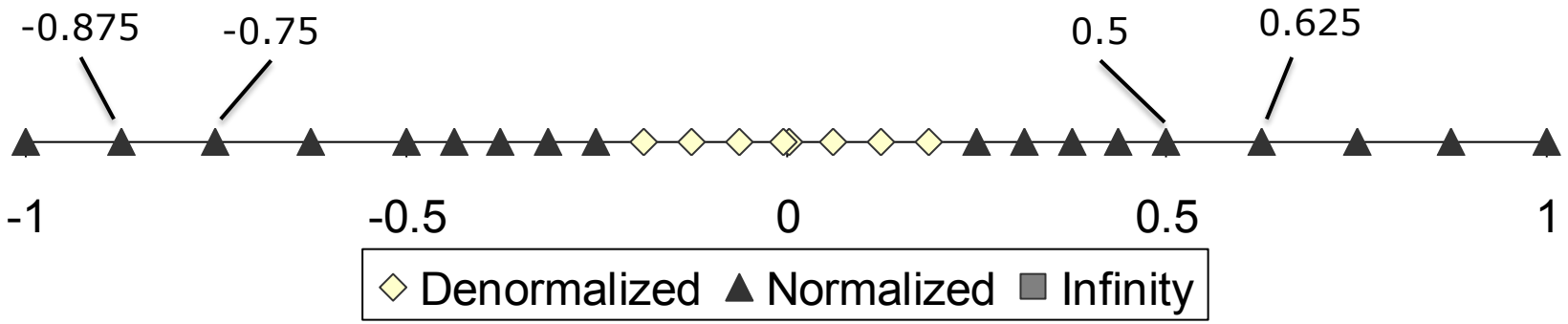



Round(-0.86) = -0.875

Round(0.55) = 0.5

# Round to nearest

Round(x) either $x_+$ or $x_-$ , whichever is nearer to x.

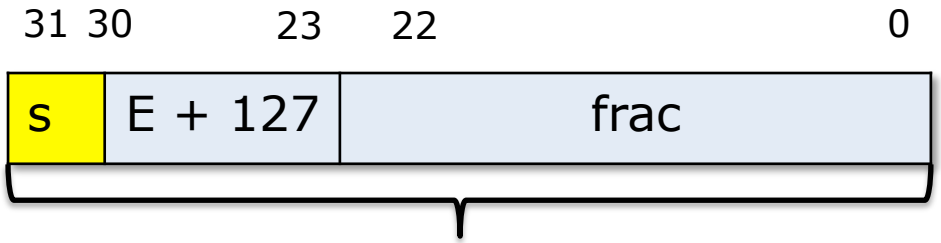



Round(-0.86) = -0.875

Round(0.55) = 0.5

In case of a tie, the one with its least significant bit equal to zero is chosen.

# single/ double precision




**single precision** (32 bits)

float f = 0.1
double d = 0.1

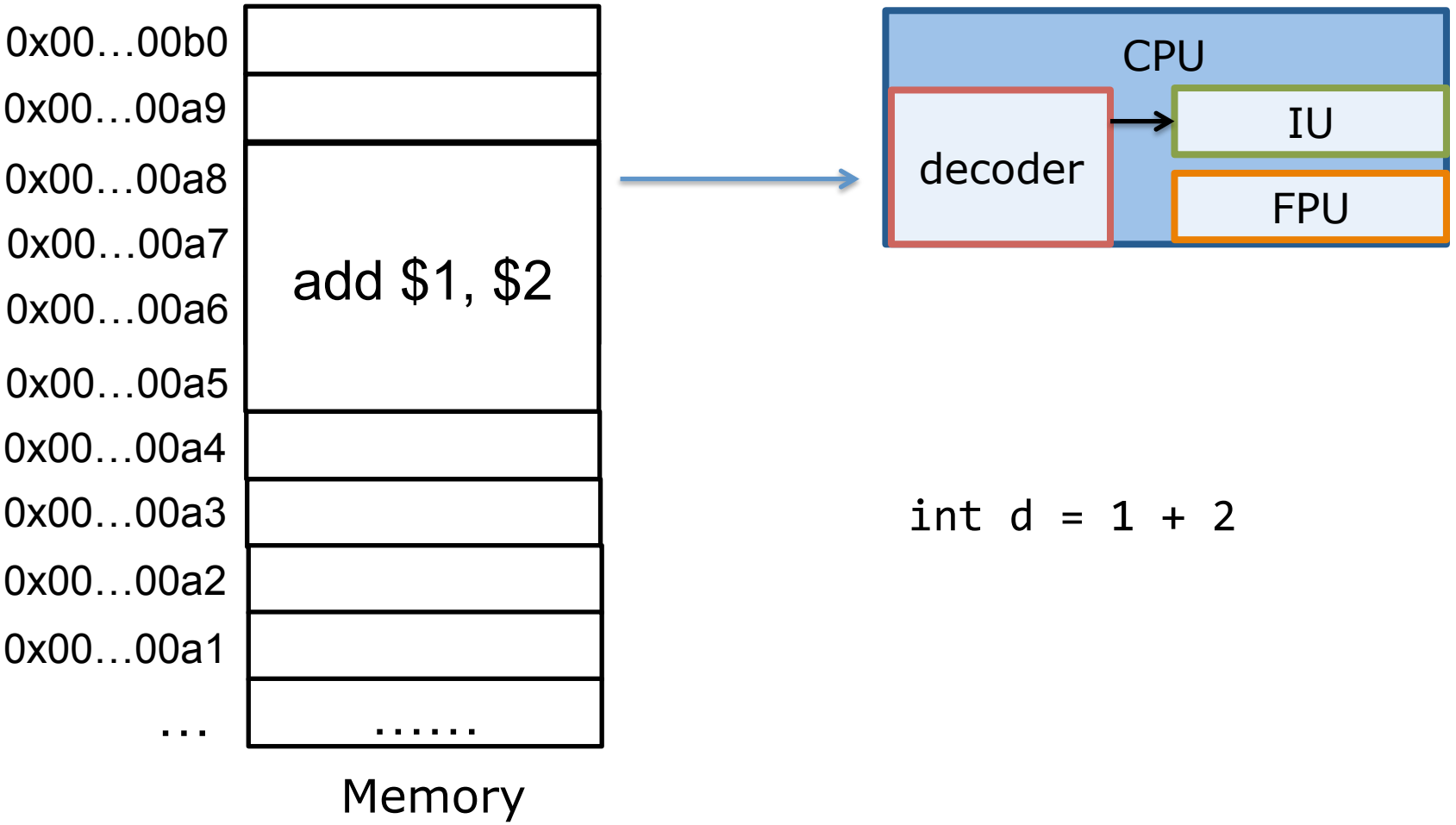

**double precision** (64 bits)

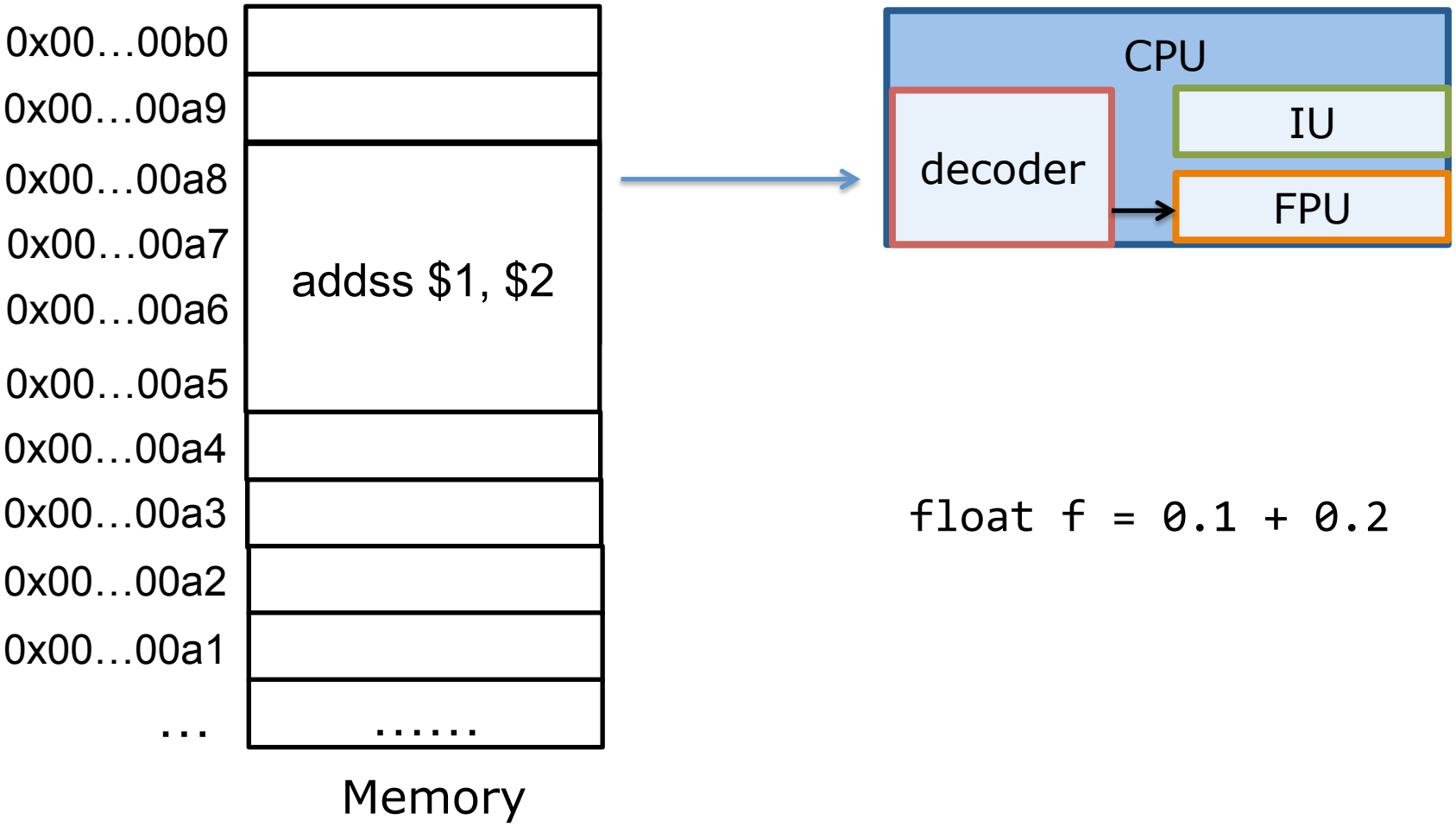
# single/ double precision

| | $E_{min}$ | $E_{max}$ | $N_{min}$ | $N_{max}$ |
|---|---|---|---|---|
| Float | -126 | 127 | $\approx 2^{-126}$ | $\approx 2^{128}$ |
| Double | -1022 | 1023 | $\approx 2^{-1022}$ | $\approx 2^{1024}$ |

# How does CPU know if it is floating point or integers ?

By having specific instruction for floating points operation.


0x00…00b0
0x00…00a9
0x00…00a8
0x00…00a7
0x00…00a6
0x00…00a5
0x00…00a4
0x00…00a3
0x00…00a2
0x00…00a1
…
add $1, $2
……
Memory
CPU
decoder
IU
FPU
int d = 1 + 2


0x00…00b0
0x00…00a9
0x00…00a8
0x00…00a7
0x00…00a6
0x00…00a5
0x00…00a4
0x00…00a3
0x00…00a2
0x00…00a1
…
addss $1, $2
……
Memory
CPU
decoder
IU
FPU
float f = 0.1 + 0.2

# First lab is out

http://news.cs.nyu.edu/~jinyang/sp18-cso/labs